Linear Programming: Model Formulation and Solution

MBA 8104: Quantitative Analysis
Chapter Topics

- Model Formulation
- A Maximization Model Example
- Graphical Solutions of Linear Programming Models
- A Minimization Model Example
- Characteristics of Linear Programming Problems
- Solving Linear Programming Problems with TORA
Objectives of business decisions frequently involve **maximizing profit** or **minimizing costs**.

Linear programming uses **linear algebraic relationships** to represent a firm’s decisions, given a business **objective**, and resource **constraints**.

**Steps in application:**
1. Identify problem as solvable by linear programming.
2. Formulate a mathematical model of the unstructured problem.
3. Solve the model.
4. Implementation
Model Components

- **Decision variables** - mathematical symbols representing levels of activity of a firm.

- **Objective function** - a linear mathematical relationship describing an objective of the firm, in terms of decision variables - this function is to be maximized or minimized.

- **Constraints** – requirements or restrictions placed on the firm by the operating environment, stated in linear relationships of the decision variables.

- **Parameters** - numerical coefficients and constants used in the objective function and constraints.
Summary of Model Formulation Steps

**Step 1**: Clearly define the decision variables

**Step 2**: Construct the objective function

**Step 3**: Formulate the constraints
Illustration 1: A Maximization Example (1 of 4)

- Product mix problem - Beaver Creek Pottery Company
- How many bowls and mugs should be produced to maximize profits given labor and materials constraints?
- Product resource requirements and unit profit:

<table>
<thead>
<tr>
<th>Product</th>
<th>Resource Requirements</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Labor (Hr./Unit)</td>
<td>Clay (Lb./Unit)</td>
</tr>
<tr>
<td>Bowl</td>
<td>1</td>
<td>4</td>
</tr>
<tr>
<td>Mug</td>
<td>2</td>
<td>3</td>
</tr>
</tbody>
</table>
LP Model Formulation

A Maximization Example (2 of 4)
LP Model Formulation

A Maximization Example (3 of 4)

Resource: 40 hrs of labor per day
Availability: 120 lbs of clay

Decision Variables:
- $x_1$ = number of bowls to produce per day
- $x_2$ = number of mugs to produce per day

Objective Function:
Maximize $Z = 40x_1 + 50x_2$
Where $Z$ = profit per day

Constraints:
1. $x_1 + 2x_2 \leq 40$ hours of labor
2. $4x_1 + 3x_2 \leq 120$ pounds of clay

Non-Negativity Constraints:
$x_1 \geq 0; x_2 \geq 0$
A Maximization Example (4 of 4)

Complete Linear Programming Model:

Maximize \[ Z = 40x_1 + 50x_2 \]

subject to:

\[ x_1 + 2x_2 \leq 40 \]
\[ 4x_2 + 3x_2 \leq 120 \]
\[ x_1, x_2 \geq 0 \]
A feasible solution does not violate any of the constraints:

Example:  
\[ x_1 = 5 \text{ bowls} \]
\[ x_2 = 10 \text{ mugs} \]
\[ Z = 40x_1 + 50x_2 = 700 \]

Labor constraint check:  
\[ 1(5) + 2(10) = 25 < 40 \text{ hours} \]

Clay constraint check:  
\[ 4(5) + 3(10) = 70 < 120 \text{ pounds} \]
An *infeasible solution* violates *at least one* of the constraints:

Example:  

\[ x_1 = 10 \text{ bowls} \]
\[ x_2 = 20 \text{ mugs} \]
\[ Z = 40x_1 + 50x_2 = 1400 \]

Labor constraint check:  
\[ 1(10) + 2(20) = 50 > 40 \text{ hours} \]
Graphical Solution of LP Models

- Graphical solution is limited to linear programming models containing *only two decision variables* (can be used with three variables but only with great difficulty).

- Graphical methods provide *visualization of how* a solution for a linear programming problem is obtained.
Maximize $Z = 40x_1 + 50x_2$ subject to:

1. $x_1 + 2x_2 \leq 40$
2. $4x_2 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Figure 2.2 Coordinates for Graphical Analysis
Maximize\: Z = 40x_1 + 50x_2 \\
subject\: to:\: \begin{align*}
1x_1 + 2x_2 & \leq 40 \\
4x_2 + 3x_2 & \leq 120 \\
x_1, x_2 & \geq 0
\end{align*}

Figure 2.3 Graph of Labor Constraint
Maximize \( Z = 40x_1 + 50x_2 \)
subject to:
\[
\begin{align*}
1x_1 + 2x_2 & \leq 40 \\
4x_2 + 3x_2 & \leq 120 \\
x_1, x_2 & \geq 0
\end{align*}
\]
Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_2 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Figure 2.5 Clay Constraint Area
Maximize $Z = 40x_1 + 50x_2$

subject to:

$1x_1 + 2x_2 \leq 40$

$4x_2 + 3x_2 \leq 120$

$x_1, x_2 \geq 0$

Figure 2.6 Graph of Both Model Constraints
Maximize $Z = 40x_1 + 50x_2$
subject to:
1. $x_1 + 2x_2 \leq 40$
2. $4x_2 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Figure 2.7  Feasible Solution Area
Objective Function Solution = $800

Graphical Solution of Maximization Model (7 of 12)

Maximize $Z = 40x_1 + 50x_2$
subject to: \[1x_1 + 2x_2 \leq 40\]
\[4x_2 + 3x_2 \leq 120\]
\[x_1, x_2 \geq 0\]

Figure 2.8 Objection Function Line for $Z = $800
Maximize \( Z = 40x_1 + 50x_2 \)

subject to:

\[
1x_1 + 2x_2 \leq 40
\]
\[
4x_2 + 3x_2 \leq 120
\]
\[
x_1, x_2 \geq 0
\]
Maximize $Z = 40x_1 + 50x_2$
subject to:
$1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

**Figure 2.10** Identification of Optimal Solution Point
Optimal Solution Coordinates

Graphical Solution of Maximization Model (10 of 12)

Maximize $Z = 40x_1 + 50x_2$
subject to:  
$1x_1 + 2x_2 \leq 40$
$4x_2 + 3x_2 \leq 120$
$x_1, x_2 \geq 0$

Figure 2.11 Optimal Solution Coordinates
Maximize $Z = 40x_1 + 50x_2$
subject to: $\begin{align*}
1x_1 + 2x_2 & \leq 40 \\
4x_2 + 3x_2 & \leq 120 \\
x_1, x_2 & \geq 0
\end{align*}$

Figure 2.12 Solutions at All Corner Points
Maximize $Z = 70x_1 + 20x_2$
subject to:  
1. $x_1 + 2x_2 \leq 40$
2. $4x_2 + 3x_2 \leq 120$
3. $x_1, x_2 \geq 0$

Figure 2.13 Optimal Solution with $Z = 70x_1 + 20x_2$
Standard form requires that all constraints be in the form of equations (equalities).

A slack variable is added to a $\leq$ constraint (weak inequality) to convert it to an equation (=).

A slack variable typically represents an unused resource.

A slack variable contributes nothing to the objective function value.
Max $Z = 40x_1 + 50x_2 + s_1 + s_2$
subject to: $1x_1 + 2x_2 + s_1 = 40$
$4x_2 + 3x_2 + s_2 = 120$
$x_1, x_2, s_1, s_2 \geq 0$

Where:

$x_1$ = number of bowls
$x_2$ = number of mugs
$s_1, s_2$ are slack variables

Figure 2.14 Solution Points A, B, and C with Slack
Illustration 2: LP Model Formulation – Minimization (1 of 8)

- Two brands of fertilizer available - Super-gro, Crop-quick.
- Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
- Super-gro costs $6 per bag, Crop-quick $3 per bag.
- Problem: How much of each brand to purchase to minimize total cost of fertilizer given following data?

<table>
<thead>
<tr>
<th>Brand</th>
<th>Chemical Contribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Nitrogen (lb/ bag)</td>
</tr>
<tr>
<td>Super-gro</td>
<td>2</td>
</tr>
<tr>
<td>Crop-quick</td>
<td>4</td>
</tr>
</tbody>
</table>
Figure 2.15 Fertilizing farmer’s field
Decision Variables:

\[ x_1 = \text{bags of Super-gro} \]
\[ x_2 = \text{bags of Crop-quick} \]

The Objective Function:

Minimize \[ Z = 6x_1 + 3x_2 \]
Where: \[ 6x_1 = \text{cost of bags of Super-Gro} \]
\[ 3x_2 = \text{cost of bags of Crop-Quick} \]

Model Constraints:

\[ 2x_1 + 4x_2 \geq 16 \text{ lb (nitrogen constraint)} \]
\[ 4x_1 + 3x_2 \geq 24 \text{ lb (phosphate constraint)} \]
\[ x_1, x_2 \geq 0 \text{ (non-negativity constraint)} \]
Minimize $Z = 6x_1 + 3x_2$
subject to:

\[ 2x_1 + 4x_2 \geq 16 \]
\[ 4x_2 + 3x_2 \geq 24 \]
\[ x_1, x_2 \geq 0 \]

Figure 2.16 Graph of Both Model Constraints
Minimize $Z = 6x_1 + 3x_2$
subject to:

- $2x_1 + 4x_2 \geq 16$
- $4x_2 + 3x_2 \geq 24$
- $x_1, x_2 \geq 0$

**Figure 2.17 Feasible Solution Area**
Minimize \( Z = 6x_1 + 3x_2 \)
subject to:
\[ 2x_1 + 4x_2 \geq 16 \]
\[ 4x_2 + 3x_2 \geq 24 \]
\( x_1, x_2 \geq 0 \)

\( x_1 = 0 \) bags of Super-gro
\( x_2 = 8 \) bags of Crop-quick
\( Z = $24 \)

\( Z = 6x_1 + 3x_2 \)

**Figure 2.18** Optimum Solution Point
A surplus variable is *subtracted from a ≥ constraint* to convert it to an equation (=).

A surplus variable *represents an excess* above a constraint requirement level.

A surplus variable *contributes nothing* to the calculated value of the objective function.

Subtracting surplus variables in the farmer problem constraints:

\[
2x_1 + 4x_2 - s_1 = 16 \text{ (nitrogen)}
\]
\[
4x_1 + 3x_2 - s_2 = 24 \text{ (phosphate)}
\]
Minimize $Z = 6x_1 + 3x_2 + 0s_1 + 0s_2$
subject to:

\[ \begin{align*}
2x_1 + 4x_2 - s_1 &= 16 \\
4x_2 + 3x_2 - s_2 &= 24 \\
x_1, x_2, s_1, s_2 &\geq 0
\end{align*} \]
For some linear programming models, the general rules do not apply.

• Special types of problems include those with:
  ▪ Multiple optimal solutions
  ▪ Infeasible solutions
  ▪ Unbounded solutions
The objective function is **parallel** to a constraint line.

Maximize \( Z = 40x_1 + 30x_2 \)
subject to:

\[
\begin{align*}
1x_1 + 2x_2 &\leq 40 \\
4x_2 + 3x_2 &\leq 120 \\
x_1, x_2 &\geq 0
\end{align*}
\]

Where:
\( x_1 = \text{number of bowls} \)
\( x_2 = \text{number of mugs} \)

**Figure 2.20** Example with Multiple Optimal Solutions
An Infeasible Problem

Every possible solution violates at least one constraint:

Maximize \( Z = 5x_1 + 3x_2 \)

subject to:

\[
\begin{align*}
4x_1 + 2x_2 & \leq 8 \\
x_1 & \geq 4 \\
x_2 & \geq 6 \\
x_1, x_2 & \geq 0
\end{align*}
\]

Figure 2.21 Graph of an Infeasible Problem
An Unbounded Problem

Value of the objective function increases indefinitely:

Maximize \( Z = 4x_1 + 2x_2 \)
subject to: \( x_1 \geq 4 \)
\( x_2 \leq 2 \)
\( x_1, x_2 \geq 0 \)

Figure 2.22 Graph of an Unbounded Problem
Characteristics of Linear Programming Problems

• A decision amongst alternative courses of action is required.
• The decision is represented in the model by decision variables.
• The problem encompasses a goal, expressed as an objective function, that the decision maker wants to achieve.
• Restrictions (represented by constraints) exist that limit the extent of achievement of the objective.
• The objective and constraints must be definable by linear mathematical functional relationships.
Properties of Linear Programming Models

- **Proportionality** - The rate of change (slope) of the objective function and constraint equations is constant.

- **Additivity** - Terms in the objective function and constraint equations must be additive.

- **Divisibility** - Decision variables can take on any fractional value and are therefore continuous as opposed to integer in nature.

- **Certainty** - Values of all the model parameters are assumed to be known with certainty (non-probabilistic).
Problem Statement: Example (1 of 3)

- Hot dog mixture in 1000-pound batches.
- Two ingredients, chicken ($3/g) and beef ($5/g).
- Recipe requirements:
  - at least 500 pounds of “chicken”
  - at least 200 pounds of “beef”
- Ratio of chicken to beef must be at least 2 to 1.
- Determine optimal mixture of ingredients that will minimize costs.
Solution: Example Problem No. 1 (2 of 3)

Step 1:
Identify decision variables.

\[ x_1 = \text{Quantity in grams of chicken in mixture} \]
\[ x_2 = \text{Quantity in grams of beef in mixture} \]

Step 2:
Formulate the objective function.

Minimize \[ Z = 3x_1 + 5x_2 \]
where \[ Z = \text{cost per 1,000-lb batch} \]
\[ 3x_1 = \text{cost of chicken} \]
\[ 5x_2 = \text{cost of beef} \]
Step 3:

Establish Model Constraints

\[ x_1 + x_2 = 1,000 \text{ lb} \]
\[ x_1 \geq 500 \text{ g of chicken} \]
\[ x_2 \geq 200 \text{ g of beef} \]
\[ x_1/x_2 \geq 2/1 \text{ or } x_1 - 2x_2 \geq 0 \]
\[ x_1, x_2 \geq 0 \]

The Model: Minimize \[ Z = 3x_1 + 5x_2 \]
subject to: \[ x_1 + x_2 = 1,000 \text{ g} \]
\[ x_1 \geq 50 \]
\[ x_2 \geq 200 \]
\[ x_1 - 2x_2 \geq 0 \]
\[ x_1, x_2 \geq 0 \]
Example Problem No. 2 (1 of 3)

Solve the following model graphically:
Maximize \( Z = 4x_1 + 5x_2 \)
subject to: \( x_1 + 2x_2 \leq 10 \)
\( 6x_1 + 6x_2 \leq 36 \)
\( x_1 \leq 4 \)
\( x_1, x_2 \geq 0 \)

Step 1: Plot the constraints as equations

Figure 2.23 Constraint Equations
Maximize $Z = 4x_1 + 5x_2$
subject to: $x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 4$
$x_1, x_2 \geq 0$

Step 2: Determine the feasible solution space

Figure 2.24 Feasible Solution Space and Extreme Points
Example Problem No. 2 (3 of 3)

Maximize $Z = 4x_1 + 5x_2$
subject to:  
$x_1 + 2x_2 \leq 10$
$6x_1 + 6x_2 \leq 36$
$x_1 \leq 4$
$x_1, x_2 \geq 0$

Step 3 and 4: Determine the solution points and optimal solution

Figure 2.25 Optimal Solution Point
Group work No. 1: Super Grain Corp. Advertising-Mix Problem

Goal: Design the promotional campaign for Crunchy Start.

The three most effective advertising media for this product are:
- Television commercials on Saturday morning programs for children.
- Advertisements in food and family-oriented magazines.
- Advertisements in Sunday supplements of major newspapers.

The limited resources in the problem are:
- Advertising budget ($4 million).
- Planning budget ($1 million).
- TV commercial spots available (5).

The objective will be measured in terms of the expected number of exposures.

Question: At what level should they advertise Crunchy Start in each of the three media?
# Cost and Exposure Data

<table>
<thead>
<tr>
<th>Cost Category</th>
<th>Each TV Commercial</th>
<th>Each Magazine Ad</th>
<th>Each Sunday Ad</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ad Budget ($4 million)</td>
<td>$300,000</td>
<td>$150,000</td>
<td>$100,000</td>
</tr>
<tr>
<td>Planning budget ($1 million)</td>
<td>90,000</td>
<td>30,000</td>
<td>40,000</td>
</tr>
<tr>
<td>Expected number of exposures</td>
<td>1,300,000</td>
<td>600,000</td>
<td>500,000</td>
</tr>
</tbody>
</table>
Group work No. 2 Think-Big Capital Budgeting Problem

Think-Big Development Co. is a major investor in commercial real-estate development projects. They are considering three large construction projects:

- Construct a high-rise office building.
- Construct a hotel.
- Construct a shopping center.

Each project requires each partner to make four investments: a down payment now, and additional capital after one, two, and three years.

Given the following table determine at what fraction should Think-Big invest in each of the three projects.
# Financial Data for the Projects

<table>
<thead>
<tr>
<th>Year</th>
<th>Office Building</th>
<th>Hotel</th>
<th>Shopping Center</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$40 million</td>
<td>$80 million</td>
<td>$90 million</td>
</tr>
<tr>
<td>1</td>
<td>60 million</td>
<td>80 million</td>
<td>50 million</td>
</tr>
<tr>
<td>2</td>
<td>90 million</td>
<td>80 million</td>
<td>20 million</td>
</tr>
<tr>
<td>3</td>
<td>10 million</td>
<td>70 million</td>
<td>60 million</td>
</tr>
</tbody>
</table>

Net present value

- Office Building: $45 million
- Hotel: $70 million
- Shopping Center: $50 million

Assume for years 0 through 3 the firm has: $25MM, $45MM, $65MM, and $80MM available. (cumulative)
A prison is trying to decide what to feed its prisoners. They would like to offer some combination of milk, beans, and oranges. Their goal is to minimize cost, subject to meeting the minimum nutritional requirements imposed by law. The cost and nutritional content of each food, along with the minimum nutritional requirements are shown below.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Niacin (mg)</td>
<td>3.2</td>
<td>4.9</td>
<td>0.8</td>
<td>13.0</td>
</tr>
<tr>
<td>Thiamin (mg )</td>
<td>1.12</td>
<td>1.3</td>
<td>0.19</td>
<td>1.5</td>
</tr>
<tr>
<td>Vitamin C (mg)</td>
<td>32.0</td>
<td>0.0</td>
<td>93.0</td>
<td>45.0</td>
</tr>
<tr>
<td>Cost ($ )</td>
<td>2.00</td>
<td>0.20</td>
<td>0.25</td>
<td></td>
</tr>
</tbody>
</table>